Approximate Optimal Atmospheric Guidance Law for Aeroassisted Plane-Change Maneuvers

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In this paper, we develop an approximate optimal guidance law for the aeroassisted plane-change problem through the use of an expansion of the Hamilton-Jacobi-Bellman equation. This guidance law maximizes the final velocity of the re-entry vehicle while meeting terminal constraints on altitude, flight-path angle, and heading angle. The expansion is made with respect to a small parameter that arises naturally as the ratio of the atmospheric scale height to the radius of the Earth. The integrable zeroth-order solution obtained when the small parameter is set to zero corresponds to a solution of the problem where the aerodynamic forces dominate the orbital forces (all nonaerodynamic forces). Higher-order terms in the expansion are determined from the solution of linear, partial differential equations requiring only quadratures. Results from this guidance law, using only a first-order correction term, are quite impressive. Comparisons are made to a numerical optimal solution and another approximate analytical solution for this problem.

I. Introduction

As shown in Ref. 1, an aeroassisted plane-change maneuver can require less fuel than a thrust-only maneuver. However, this requires the development of efficient control laws to fully utilize the aerodynamic forces. Because of the nonlinearity of the dynamics, an analytical solution without approximation is impossible. Even the numerical solution of the optimization problem is a difficult task, exacerbated by the limitations of present onboard computer technology. Therefore, we choose to explore approximation techniques that result in analytical solutions or greatly reduced computations in obtaining the guidance laws. One such technique used in developing an analytical control law involves the approximation that the orbital forces are negligible when compared to the aerodynamic forces, or are included through the approximation of Loh's constant.[2-4] The object of this paper is to investigate an expansion procedure and apply it to the development of an approximate optimal atmospheric guidance law for the aeroassisted plane-change maneuver.

This approximate optimal guidance law is determined by an expansion of the Hamilton-Jacobi-Bellman equation with respect to a small parameter.[5] The small parameter for this problem arises simply as the ratio of the scale height (used in the exponential density assumption) to the radius of the Earth. This choice of a small parameter implies that for the zeroth-order problem the aerodynamic forces dominate the orbital forces.[2] As a direct consequence, the zeroth-order problem is integrable, resulting in an analytic zeroth-order control law.[2] The higher-order terms in the expansion are determined from first-order linear partial differential equations whose characteristic curves are determined by the zeroth-order solution. Although these equations cannot be solved in closed form, their solution can be obtained from the evaluation of simple quadratures. Therefore, a feedback control law is given by numerical integrations easily performed onboard. For a discussion of the Hamilton-Jacobi-Bellman expansion approach, see Sec. II.

This approach, applied in Sec. III, is similar to that given in Ref. 6, except that an alternate variable is identified as the independent variable instead of flight-path angle. This new variable simplifies the solution procedure and allows the use of the theoretical results in Ref. 6. As the results in Sec. IV show, impressive performance of the guidance law is obtained.

II. Perturbed Hamilton-Jacobi-Bellman Equation

Breakwell et al.[5] obtained a uniformly valid feedback control law in the form of a power series in a small parameter for a certain class of optimal control problems by solving the Hamilton-Jacobi-Bellman partial differential equation (HJB-PDE). The optimization problem of interest is subject to nonlinear dynamics and terminal constraints, and has a performance index composed of the terminal states. Although the intent of Ref. 5 was to solve a singular perturbation problem, it inspired the use of the HJB-PDE to solve the ordinary perturbation aeroassisted maneuver problem.

The dynamic system is given by

$$\dot{y} = f(y,u) + \epsilon g(y), \qquad y(t) = x = \text{given} \tag{1}$$

where y is an n-dimensional state vector, u is an m-dimensional control vector, ϵ is a small parameter, τ is the independent variable, $\dot{y} \triangleq \mathrm{d}y/\mathrm{d}\tau$, x is the initial state, and t is the initial value of the independent variable. The n-vector functions $f(y,u,\tau)$ and $g(y,\tau)$ are assumed analytic in the region of interest with respect to their arguments. Note that g is not a function of the control vector u. As shown in Appendix A, it is not generally necessary to assume this form. However, our system has this form, which allows for simplification in the solution of the problem.

The optimization problem is as follows: Find u to minimize $J = \phi(y_f)$, subject to the dynamics of Eq. (1) and the terminal constraint $\psi(y_f) = 0$. The Hamilton-Jacobi-Bellman partial differential equation is

$$P_t = -H^{\text{opt}} \tag{2a}$$

$$H^{\text{opt}} = \min_{u \in \mathcal{U}} H = P_x\,(f^{\text{opt}} + \epsilon g) \tag{2b}$$

where $\mathcal{U}$ is the class of continuous-bounded controls, $f^{\text{opt}} \equiv f[y,u,{}^{\text{opt}}(x,t),\tau]$, and $u^{\text{opt}}(x,t)$ is given by the optimality condition $H_u = 0$, assuming that H_{uu} is positive definite.[7]

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$P(x,t)$ is the optimal return function defined as the optimal value of the performance index for an optimal path starting at x and t and satisfying the terminal conditions.

We express $P(x,t)$ as a series expansion in ϵ as

$$P(x,t) = \sum_{j=0}^{\infty} P_j(x,t)\epsilon^j \tag{3}$$

and the optimal control as

$$u^{\text{opt}}(x,t) = \underline{u}^{\text{opt}}(x,P_x,t) = \sum_{j=0}^{\infty} u_j(x,t)\epsilon^j \tag{4}$$

where the expansion for $u^{\text{opt}}(x,t)$ is obtained by substituting Eq. (3) into $\underline{u}^{\text{opt}}(x,P_x,t)$ in Eq. (4) and expanding the function. Hence, by determining the partials P_{i_x}, we are able to construct an optimal control law in a feedback form. The zeroth-order term u_0 is the optimal control for the reduced-order problem $\epsilon = 0$. If we can find an analytic solution for u_0 (as found in the aeroassisted plane-change maneuver), then the higher-order solutions are determined by the expansion of the HJB-PDE

$$P_t = \sum_{j=0}^{\infty} P_{j_t}\epsilon^j = -\left(\sum_{j=0}^{\infty} P_{j_x}\epsilon^j\right)\left(\epsilon g + \sum_{j=0}^{\infty} f_j\epsilon^j\right)$$

where the f_j are the coefficients of the Taylor series expansion of $f(y,u)$ as shown in Appendix A.

Here, we show that this expansion leads to first-order, linear partial differential equations for P_j as

$$\begin{aligned} P_{j_t} + P_{j_x} f_0^{\text{opt}} &= -P_{j-1_x} g - \sum_{n=0}^{j-1} P_{n_x} f_{j-n} \\ &= R_j(x,t,P_{j-1},\ldots,P_0), \qquad j = 1,\ldots \end{aligned} \tag{5}$$

with the boundary condition $P_j(x_f,t_f) = 0$, $j = 1,\ldots$. f_0 is the dynamics of the zeroth-order problem ($\epsilon = 0$) with the control $u = u_0$. The forcing term R_j is a function of the lower-order terms of P. This procedure is carried out explicitly in Appendix A where the first- and second-order terms of R_j are expressed as

$$R_1 = -P_{0_x} g(x)$$

$$R_2 = \frac{u_1^2}{2} P_{0_x} f_{uu}(x,u_0) - P_{1_x} g(x)$$

Partial differential equations of this type are solved by the method of characteristics.[8] The characteristic curves of the equations, for any order term of P_j, are given by the zeroth-order optimal trajectory

$$\dot{y}_0 = f_0$$

whose solution is denoted as $y_0(\tau; x,t)$. Then, the solution for P_j in Eq. (5) is given by

$$P_j(x,t) = -\int_t^{t_f} R_j \, \mathrm{d}\tau \tag{6}$$

where

$$R_j = R_j[y_0,\tau,P_{j-1}(y_0,\tau),\ldots, P_0(y_0,\tau)], \qquad j = 1,\ldots$$

along the zeroth-order path.

The partials P_{j_x}, which are needed to construct the optimal control u_j, are given by differentiating Eq. (6) with respect to the arbitrary current conditions x as follows:

$$P_{j_x} = \frac{\partial P_j}{\partial x} = -\int_t^{t_f} \frac{\partial R_j}{\partial x}\, \mathrm{d}\tau - R_j\bigg|_{t_f} \frac{\partial t_f}{\partial x} \tag{7}$$

Note that the higher-order corrections for the control only depend upon the zeroth-order state solution. Therefore, when using this solution as a guidance law the approximate higher-order state trajectories need not be evaluated.

III. Aeroassisted Plane-Change Maneuver

Application of the preceding feedback expansion technique involves two major steps. The solution of the optimal control problem is actually the last step. The first step is finding the appropriate form of the dynamical equations through the prudent choice of system variables.

A. Dynamical Equations

The equations of motion for a point mass flying about a spherical rotating planet are given as follows[9-11]:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = V \sin\gamma \tag{8a}$$

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} = -\frac{D}{m} - g \sin\gamma + r\omega_p^2 F_1 \tag{8b}$$

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\tau} = \frac{L}{mV}\cos\mu - \left(\frac{g}{V} - \frac{V}{r}\right)\cos\gamma + 2E_2\omega_p + \frac{r\omega_p^2}{V} F_2 \tag{8c}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = \frac{L \sin\mu}{mV\cos\gamma} + \frac{V}{r}\cos\gamma \cos\psi \tan\phi + \frac{2E_3\omega_p + (r\omega_p^2/V)F_3}{\cos\gamma} \tag{8d}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{V\cos\gamma \sin\psi}{r} \tag{8e}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{V\cos\gamma \cos\psi}{r\cos\phi} \tag{8f}$$

where r is the distance from the center of the planet to the center of mass of the vehicle, V is the velocity, γ is the flight-path angle, ψ is the heading angle, ϕ is the crossrange angle, θ is the downrange angle, τ is the time, m is the mass of the vehicle, μ is the bank angle, and g is the local gravitational acceleration.[6] The functional forms for F_1, F_2, F_3, E_2, and E_3 are given as follows[9-11]:

$$E_2 = \cos\phi \cos\psi \tag{9a}$$

$$E_3 = \cos\gamma \sin\phi - \sin\gamma \cos\phi \sin\psi \tag{9b}$$

$$F_1 = \cos\phi(\sin\gamma \cos\phi - \cos\gamma \sin\phi \sin\psi) \tag{9c}$$

$$F_2 = \cos\phi(\cos\gamma \cos\phi + \sin\gamma \sin\phi \cos\psi) \tag{9d}$$

$$F_3 = \sin\phi \cos\phi \cos\psi \tag{9e}$$

B. Transformed Dynamics

The preceding dynamical equations (8) do not exhibit an apparent separation into the primary and perturbation portion required by the Hamilton-Jacobi-Bellman expansion technique discussed in Sec. II and shown in Eq. (1). However, this separation can be obtained through the use of various transformations and modeling assumptions. The transformations and assumptions used to obtain a suitable set of separable dynamical equations are presented here.

The dynamical equations can be transformed through the use of the following nondimensional variables:

$$w = \frac{C_L^* \rho S \beta}{2m}, \qquad v = \ln\left(\frac{V^2}{gr}\right)$$

where w and v are nondimensional altitude and velocity, respectively.

In addition to a change of variables, some modeling assumptions are also made. First, an inverse square gravitational field is assumed:

$$g = \frac{\bar{\mu}}{r^2} \tag{10}$$

where $\bar{\mu}$ is the planetary gravitational constant. A locally exponential atmospheric density model is assumed:

$$\rho = \rho_r e^{-(r/\beta)} \tag{11}$$

where β is the exponential atmospheric scale height and ρ_r is the reference atmospheric density.

To further simplify the equations, the following aerodynamic force modeling relationships are assumed:

$$L = \tfrac{1}{2}\rho V^2 S C_L, \qquad D = \tfrac{1}{2}\rho V^2 S C_D$$

$$C_D = C_{D_0} + K C_L^2 \qquad \text{(parabolic drag polar)}$$

where L is the lift, D is the drag, C_D is the drag coefficient, S is the vehicle aerodynamic surface area, C_L is the lift coefficient, C_{D_0} is the zero-lift drag coefficient, and K is the parabolic drag model constant. In addition, λ is the normalized lift coefficient and E^* is the maximum lift-to-drag ratio:

$$\lambda = C_L / C_L^* \tag{12}$$

$$E^* = C_L^* / C_D^* \tag{13}$$

Also, the following nondimensional lift-force δ and side-force σ control variables are defined:

Lifting force

$$\delta = \lambda \cos\mu$$

Side force

$$\sigma = \lambda \sin\mu$$

Finally, the flight-path angle γ is assumed to be small; therefore, $\cos\gamma \approx 1$ and $\sin\gamma \approx \gamma$.

In addition to the preceding choices of nondimensional transform variables, a new independent variable or time scale is chosen. By letting

$$\frac{d\tau}{dz} = \frac{\beta}{wV} = \frac{mV}{L^*} \tag{14}$$

z becomes the new monotonically increasing independent variable.

Finally, an appropriate small expansion parameter is identified. This small parameter must multiply the perturbing portions of the dynamical equations. This will result in a zeroth-order set of equations free of the undesired perturbing effects. Also, the solution to the resulting zeroth-order problem must be integrable. The small parameter ϵ is chosen to be the ratio of the atmospheric scale height β and mean planetary radius r_s:

$$\epsilon = \beta / r_s \tag{15}$$

These modeling assumptions and transformations result in the following set of separable dynamical equations:

$$\frac{dw}{dz} = -\gamma \tag{16a}$$

$$\frac{dv}{dz} = -\frac{(1+\sigma^2+\delta^2)}{E^*} - \epsilon\frac{r_s}{rw}\left[(2e^{-v}-1)\gamma - 2\left(\frac{V_w}{V}\right)^2 F_1\right] \tag{16b}$$

$$\frac{d\gamma}{dz} = \delta + \epsilon\frac{r_s}{rw}\left[(1-e^{-v}) + 2\left(\frac{V_w}{V}\right)C_2 + \left(\frac{V_w}{V}\right)^2 F_2\right] \tag{16c}$$

$$\frac{d\psi}{dz} = \sigma - \epsilon\frac{r_s}{rw}\left[\cos\psi\tan\phi + 2\left(\frac{V_w}{V}\right)C_3 + \left(\frac{V_w}{V}\right)^2 F_3\right] \tag{16d}$$

$$\frac{d\phi}{dz} = \epsilon\left[\frac{r_s\sin\psi}{rw}\right] \tag{16e}$$

$$\frac{d\theta}{dz} = \epsilon\left[\frac{r_s\cos\psi}{rw\cos\phi}\right] \tag{16f}$$

where $V_w = r\,\omega_p$ is the relative wind velocity due to the rotation of the planet and ω_p is the planetary rotation rate.

The benefit of using this new variable z is to simplify the equations and prevent the control from appearing in the perturbation dynamics. Other choices of independent variable (ψ for instance) would introduce the control into the perturbation dynamics. Finally, the specified choice of small parameter eliminates the orbital portions of the dynamical equations while keeping the aerodynamic terms. As will be shown in following sections, the resulting zeroth-order equations can be solved analytically. Therefore, this choice of small parameter satisfies the stated requirements.

C. Optimal Control Problem

We now present the solution of the optimal control problem for the dynamical system represented in Eq. (16) using the theory developed in Sec. II. For this discussion a nonrotating planet will be assumed ($\omega_p = 0$). This does not change the solution of the zeroth-order problem since ω_p does not enter into the primary dynamics. Furthermore, the addition of rotational terms in the perturbation dynamics only serves to complicate the formulation of the quadratures through the presence of additional partial derivatives. Other than that, the process remains essentially the same. However, some results for a nonzero rotation rate will be briefly discussed in the results section.

1. *Optimal Control Relationships*

The optimal control relationships for the aeroassisted plane-change problem are obtained by finding the controls that minimize the negative of the final velocity subject to Eq. (16) and the prescribed boundary conditions.

Initial Conditions:

$$\psi_i,\ w_i,\ v_i,\ \text{and}\ \gamma_i \equiv \text{given}$$

Terminal Condition:

$$\psi_f,\ w_f \equiv \text{given}$$

$$\gamma_f \equiv \text{given or free}$$

The Hamiltonian becomes

$$H = P_\psi\sigma - P_w\gamma - P_v\left[\frac{(1+\sigma^2+\delta^2)}{E^*} + \epsilon\left(\frac{r_0(2e^{-v}-1)\gamma}{wr}\right)\right] + P_\gamma\left[\delta + \epsilon\left(\frac{r_0(1-e^{-v})}{wr}\right)\right]$$

where the partial derivatives P_ψ, P_w, P_v, and P_γ can be interpreted as Lagrange multipliers.[6] This Hamiltonian leads to the following equations for the optimal control in terms of the Lagrange multipliers:

$$\delta = \frac{E^* P_\gamma}{2P_v}, \qquad \sigma = \frac{E^* P_\psi}{2P_v} \tag{17}$$

2. *Zeroth-Order Optimal Control*

Now that we have the optimal control relationships in terms of the Lagrange multipliers, we can apply the feedback expansion technique discussed in Sec. II. As stated in Sec. II, this technique depends on finding an analytical solution to the zeroth-order problem. For this problem, the zeroth-order problem is analogous to assuming that the aerodynamic forces dominate the orbital forces. The zeroth-order dynamical equations ($\epsilon = 0$) are

$$\frac{dw}{dz} = -\gamma$$

$$\frac{dv}{dz} = \frac{-(1+\sigma^2+\delta^2)}{E^*}$$

$$\frac{d\gamma}{dz} = \delta$$

$$\frac{d\psi}{dz} = \sigma$$

with the same prescribed boundary conditions as the unreduced problem.

By solving this optimal control problem, the following zeroth-order optimal control relationships are obtained:

$$\delta_0 = \frac{-E^*P_{0_\gamma}}{2}, \qquad \sigma_0 = \frac{-E^*P_{0_\psi}}{2} = \text{const}$$

As shown in Ref. 4 and in Appendix B for this problem, the solution for the zeroth-order optimal side force σ_0 can be found from the solution of a fourth-order polynomial in σ_0:

$$A_4\sigma_0^4 + A_3\sigma_0^3 + A_2\sigma_0^2 - A_0 = 0$$

where

$$A_4 = 36\,(\Delta w)^2$$

$$A_3 = 24\,(\gamma_f + \gamma_i)\,\Delta w \Delta\psi$$

$$A_2 = [(\Delta\psi)^2 + (\Delta\gamma)^2 + 3\,(\gamma_f + \gamma_i)^2]\,(\Delta\psi)^2$$

$$A_0 = (\Delta\psi)^4$$

The zeroth-order optimal lifting force δ_0 is given by

$$\delta_0 = \frac{-E^*}{2}\,(P_{0_w}z + C)$$

where P_{0_w} and C are constants associated with the solution of the zeroth-order optimal control problem.

The zeroth-order lift coefficient λ_0 and bank angle μ_0 are now given by

$$\lambda_0 = C_L/C_L^* = \sqrt{(\sigma_0^2 + \delta_0^2)}$$

$$\mu_0 = \tan^{-1}\left(\frac{\sigma_0}{\delta_0}\right)$$

3. *Control Expansion*

Once the zeroth-order problem is solved, we can begin the procedure of building the approximate optimal control using the zeroth-order control and adding higher-order correction terms. These correction terms are constructed from the optimal control relationships, Eq. (17), and expansions of the Lagrange multipliers in the following manner.

The Lagrange multipliers can be expanded in the small parameter ϵ as follows:

$$P_\gamma = P_{0_\gamma} + P_{1_\gamma}\epsilon + P_{2_\gamma}\epsilon^2 + P_{3_\gamma}\epsilon^3 + \ldots$$

$$P_\psi = P_{0_\psi} + P_{1_\psi}\epsilon + P_{2_\psi}\epsilon^2 + P_{3_\psi}\epsilon^3 + \ldots$$

$$P_v = P_{0_v} + P_{1_v}\epsilon + P_{2_v}\epsilon^2 + P_{3_v}\epsilon^3 + \ldots$$

Similarly, the controls σ and δ can also be expanded

$$\sigma = \sigma_0 + \sigma_1\epsilon + \sigma_2\epsilon^2 + \sigma_3\epsilon^3 + \ldots$$

$$\delta = \delta_0 + \delta_1\epsilon + \delta_2\epsilon^2 + \delta_3\epsilon^3 + \ldots$$

Substituting the previous expansions into the control relationships of Eq. (17) and equating terms of equal powers of ϵ yields the following feedback-control expansion terms:

$$\sigma_0 = \frac{E^*P_{0_\psi}}{2P_{0_v}}$$

$$\sigma_1 = \sigma_0\left(\frac{P_{1_\psi}}{P_{0_\psi}} - \frac{P_{1_v}}{P_{0_v}}\right)$$

$$\vdots$$

$$\delta_0 = \frac{E^*P_{0_\gamma}}{2P_{0_v}}$$

$$\delta_1 = \delta_0\left(\frac{P_{1_\gamma}}{P_{0_\gamma}} - \frac{P_{1_v}}{P_{0_v}}\right)$$

$$\vdots$$

where the zeroth-order optimal solutions correspond to zeroth-order terms.

The higher-order corrections are obtained by solving for the higher-order Lagrange multiplier terms and substituting back into the preceding equations. The Lagrange multiplier terms are obtained from the quadratures represented by Eqs. (6) and

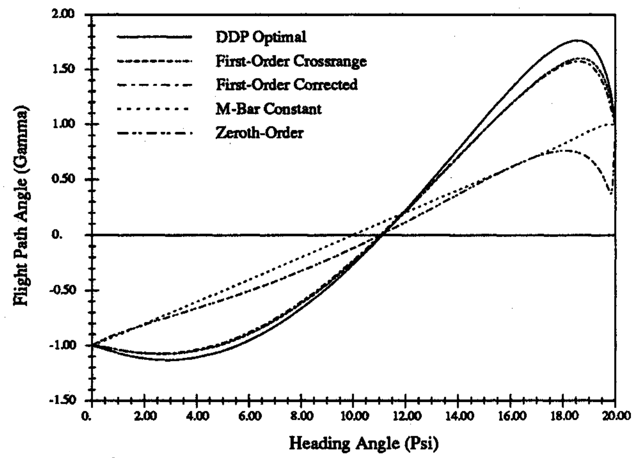

Fig. 1 Flight-path angle γ comparison plot for 20 deg.

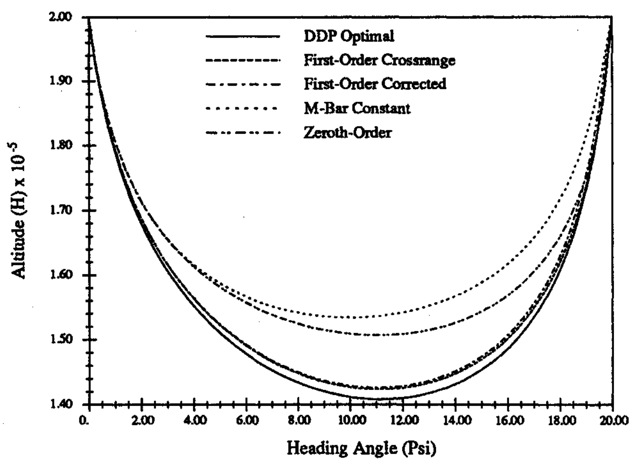

Fig. 2 Altitude H comparison plot for 20 deg.

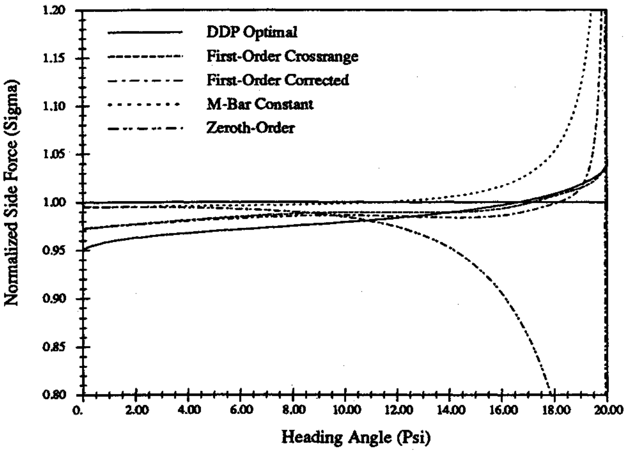

Fig. 3 **Side force σ comparison plot for 20 deg.**

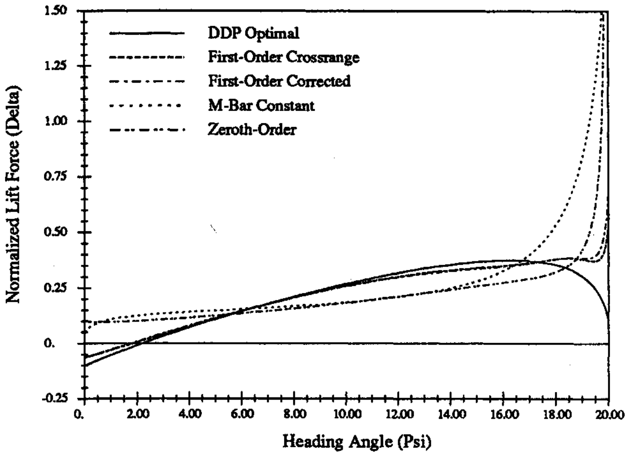

Fig. 4 **Lift force δ comparison plot for 20 deg.**

(7) and which require solution by numerical integration. This is done in Appendix C for the first-order correction terms.

D. Numerical Results

To verify the expansion approach for use in solving the aerodynamic plane-change problem, a number of simulations were run. In each of the simulations, one technique, of a number of various techniques, is used to calculate a new control for each integration point along the trajectory. In this way, the techniques are used as *feedback control algorithms.* The simulation program is written in Fortran and run on a Control Data Cyber Dual 170/750 computer. The Differential Dynamic Programming (DDP) optimal solutions are generated on an Apple Macintosh II using a Basic program written by Dr. Bernt Jämark. Unfortunately, only solutions for the nonrotating case can be generated with this program.

The model used in the simulations is the Maneuverable Research Re-entry Vehicle (MRRV) from Ref. 4. The initial conditions for all simulations are $H_i = 200{,}000$ ft, $\gamma_i = -1.00$ deg, $V_i = 25{,}945$ ft/s, and $\psi_i = \phi_i = \theta_i = 0.0$ The specified terminal end-point conditions are $H_f = 200{,}000$ ft, $\gamma_f = 1.00$ deg, and $\psi_f = 10$, 20, 30, and 40 deg. The mean planetary surface is $r_s = 20{,}926{,}428$ ft and the planetary gravitational constant is $\bar{\mu} = 1.40895 \times 10^{16}$ ft^3/s^2. The exponential atmospheric scale height is $\beta = 21{,}250$ ft. For the simulations in which planetary rotational effects are included in the dynamical model, $\omega_p = 0.00416$ deg/s. Finally, the aerodynamic parameters for the MRRV are $K = 1.4$ and $C_{D_0} = 0.032$.

The results presented here compare five solutions using three separate methods. The first solution is generated using Differential Dynamic Programming (DDP), a second order sweep method.[12] This provides a "true" optimal solution with which to compare the approximate solutions. The four approximate solutions are generated using two separate techniques. One solution is generated by using the Loh's constant ($\bar{M}$) approximation method from Refs. 3 and 13. The three remaining solutions are generated using the expansion technique discussed in Sec. 2. The first is a zeroth-order solution containing no corrections for any perturbing effects and is employed for 10- and 20-deg plane changes. Unfortunately, the zeroth-order solution is unable to meet the end-conditions for plane changes greater than 20 deg. The final two solutions are first-order corrected solutions. One solution neglects the effects of crossrange (first-order corrected); the final solution includes corrections for crossrange (first-order crossrange).

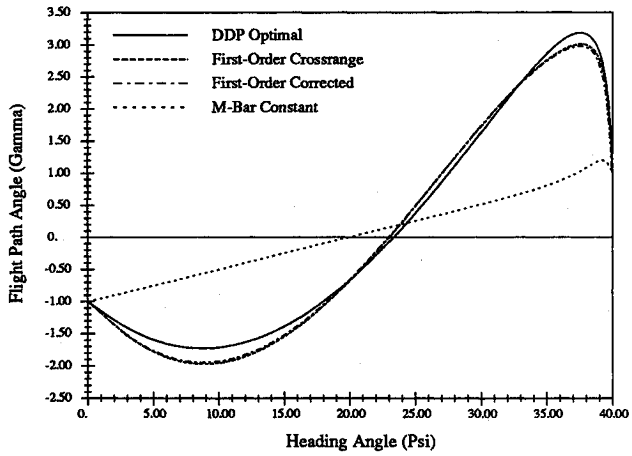

Fig. 5 **Flight-path angle γ comparison plot for 40 deg.**

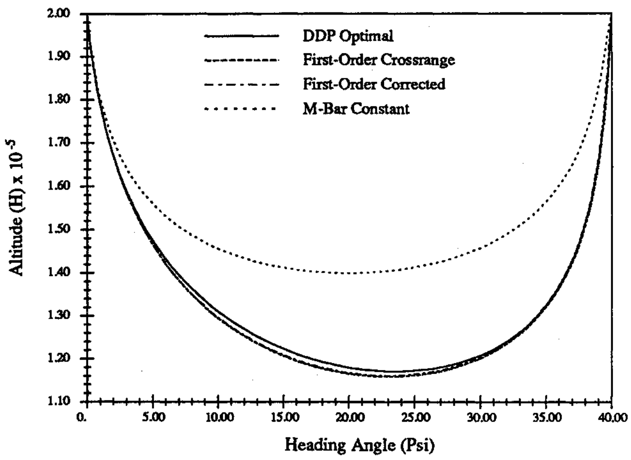

Fig. 6 **Altitude H comparison plot for 40 deg.**

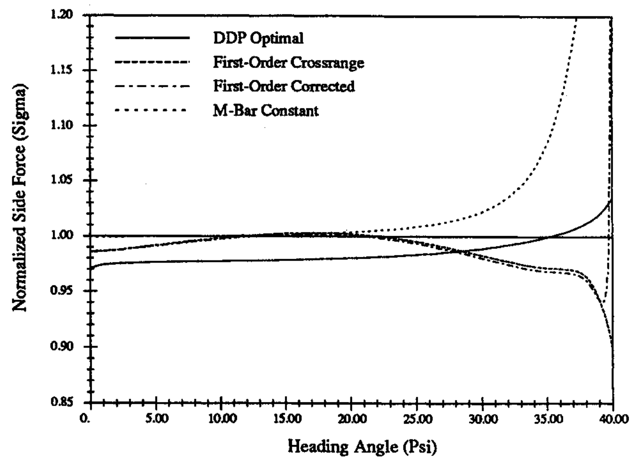

Fig. 7 **Side force σ comparison plot for 40 deg.**

Table 1 Comparison of final velocities[a]

Case	10 deg	20 deg	30 deg	40 deg
DDP Optimal	24011	22273	20661	19166
Crossrange	24009	22267	20651	19147
First order	24008	22263	20647	19143
$\bar{M}$ constant	23990	22129	20282	18455
Zeroth order	23949	21622	___	___

[a]Final velocity results, ft/s.

Table 2 Comparison of the final velocity differences[a]

Case	10 deg	20 deg	30 deg	40 deg
Crossrange	2	6	10	19
First order	3	10	14	23
$\bar{M}$ constant	21	144	379	711
Zeroth order	62	651	___	___

[a]Change from optimal, ft/s.

Table 3 Comparison of final velocities with rotation[a]

Case	V-final	ΔV-final
Crossrange and rotation	22294	33
Rotation	22284	23
Crossrange	22273	12
First order	22261	0
Zeroth order	21980	−281
$\bar{M}$ Constant	___	___

[a]Final velocity results: 20-deg plane change, ft/s.

Solution comparisons of state and control histories for flight-path angle γ, altitude H, normalized side force σ, and normalized lift force δ can be seen in Figs. 1-8 for 20- and 40-deg plane changes. The states γ and H and the controls σ and δ are used in the comparisons since they give a good indication of the character of the respective solutions. Finally, results are given in Table 1 for the final velocity (the performance index), and in Table 2 for the difference in final velocity from the DDP optimal solution.

The results of the simulations represented in Figs. 1-8 and Tables 1 and 2 are indications of the performance of the various solution techniques with respect to the DDP optimal solution. As exhibited in the tables, one good indication of performance is the comparison of final velocities. These velocity comparisons indicate an overall deterioration in performance for increasing plane change. However, there is a large difference between solutions in the extent of deterioration.

The zeroth-order solution will not meet the end-point conditions for plane changes in excess of 20 deg. The $\bar{M}$-constant solution is an improvement over the zeroth-order solution and can achieve plane changes in excess of 40 deg, but not without a severe penalty in terminal velocity. The first-order corrected solutions perform much better with respect to the DDP solution. The simple first-order corrected solution neglecting crossrange effects has a marked improvement in performance and is very close to the DDP solution. The crossrange corrected solution is the best, but only with a slight improvement over the simple first-order corrected solution.

These same characteristics can be seen in the plots as well. The zeroth-order solution deviates sharply from the DDP solution for a 20-deg plane change. The $\bar{M}$-constant solution also deviates sharply but still manages to meet the end-point conditions for all plane changes. Both of the first-order corrected solutions follow the DDP optimal solution rather closely. In fact, the first-order corrected solution including crossrange effects barely differs from the first-order corrected solution without crossrange effects.

Some simulations were also run where rotational effects are included in the dynamical model. As stated previously, this does not change the solution procedure; it merely adds to the number of partial derivatives required in the evaluation of the quadratures. Unfortunately, no DDP optimal solution was

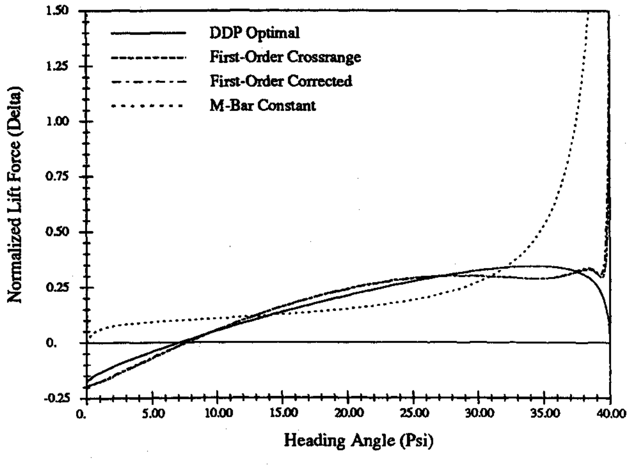

Fig. 8 Lift force δ comparison plot for 40 deg.

available for comparison; therefore, comparisons are made with respect to the simple first-order corrected solution. This can be seen in Table 3 where final velocity results are compared for a 20-deg plane change. These rotational results exhibit the same performance characteristics as the nonrotational results. However, the rotational correction terms begin to deteriorate more rapidly as we proceed to larger plane changes.

IV. Conclusions

An approximate optimal guidance law has been derived and applied to the aeroassisted plane-change problem. The performance of this new scheme is shown to be quite impressive in comparison to a previous scheme and a numerically generated optimal path. Although this scheme requires that quadrature integration be performed at each sample time at which the control is calculated, complex numerical iterative optimization schemes are avoided as a basis for optimal control. Finally, the expansion is based on a physical parameter that is clearly small.

If this technique were to be applied as a real-time guidance scheme, some additional questions need to be addressed. Most of these issues arise from the numerical aspects of solving the quadratures that arise from the solution of the higher-order correction terms: computational load and throughput time, sensitivity to integration step size, sensitivity to sample rate, and sensitivity to computational delay. In some cases, it may be possible to obtain closed-form or approximate closed-form solutions to the quadratures and eliminate the problem altogether. The sensitivity to computational delay is really a nonproblem since the entire optimal solution is obtained; therefore a solution can be projected forward to account for any delays in the computations.

There are a number of issues associated with the Hamilton-Jacobi-Bellman expansion technique that are beyond the scope of this paper. Many of the mathematical issues associated with this technique are addresses in some detail in Ref. 14; this includes issues associated with convergence. The calculus-of-variation approach is also addressed.[14] Additional topics of future research involve the inclusion of modeling uncertainties and stochastic uncertainties in the dynamical system. Finally, additional research is needed in order to include state and inequality constraints on the optimization problem.

Appendix A: Hamilton-Jacobi-Bellman Expansion Terms

The solution of the optimal control problem requires evaluation of the Lagrange multipliers. The evaluation of the Lagrange multipliers P_{j_x} require the solution of quadrature integrals that are functions of the function R_j. The evaluation of R_j, Eq. (5), requires the previously evaluated Lagrange multipliers (P_{k_x}) and the coefficients f_{k-1} and g_k for $k = 1,\ldots, j-1$. f_j and g_j are the coefficients of the jth term in the ϵ-power series expansion of f and g, i.e.,

$$f(y,u^{\text{opt}}) = \sum_{j=0}^{\infty} f_j \epsilon^j \tag{A1}$$

$$g(y,u^{\text{opt}}) = \sum_{j=0}^{\infty} g_j \epsilon^j \tag{A2}$$

Since f and g are assumed to be sufficiently differentiable, f and g are expressible in a power series in ϵ about $\epsilon = 0$. This yields

$$f = \sum_{j=0}^{\infty}\left[\left(\frac{1}{j!}\frac{\mathrm{d}^j f(y,u)}{\mathrm{d}u^j}\bigg|_{\epsilon=0}\right)\left(\sum_{k=1}^{\infty} u_k\,\epsilon^k\right)^j\right] \tag{A3}$$

$$g = \sum_{j=0}^{\infty}\left[\left(\frac{1}{j!}\frac{\mathrm{d}^j g(y,u)}{\mathrm{d}u^j}\bigg|_{\epsilon=0}\right)\left(\sum_{k=1}^{\infty} u_k\,\epsilon^k\right)^j\right] \tag{A4}$$

Using these relationships, the first four terms of Eq. (A3) and (A4) are

$$f_0 = f(y,u_0) \tag{A5}$$

$$f_1 = u_1 f_u(y,u_0) \tag{A6}$$

$$f_2 = \frac{u_1^2}{2} f_{uu}(y,u_0) + u_2 f_u(y,u_0) \tag{A7}$$

$$f_3 = \frac{u_1^3}{6} f_{uuu}(y,u_0) + u_1u_2 f_{uu}(y,u_0) + u_3 f_u(y,u_0) \tag{A8}$$

$$g_0 = g(y,u_0) \tag{A9}$$

$$g_1 = u_1 g_u(y,u_0) \tag{A10}$$

$$g_2 = \frac{u_1^2}{2} g_{uu}(y,u_0) + u_2 g_u(y,u_0) \tag{A11}$$

$$g_3 = \frac{u_1^3}{6} g_{uuu}(y,u_0) + u_1u_2 g_{uu}(y,u_0) + u_3 g_u(y,u_0) \tag{A12}$$

where for notational simplicity (y,u_0) will be assumed but not explicitly denoted in the remaining equations in this Appendix. Note that for the general case, f_j and g_j depend on the u_j determined from the optimality condition:

$$P_x[f_u + \epsilon g_u] = \left(\sum_{j=0}^{\infty} P_{j_x}\epsilon^j\right)\left(\sum_{j=0}^{\infty}(f_{j_u} + \epsilon g_{j_u})\epsilon^j\right) = 0 \tag{A13}$$

Now by substituting Eqs. (A5–A12) into Eq. (A13), multiplying out, and equating like powers of ϵ, the following relationships are obtained:

$$\epsilon^0 \;:\; P_{0_x} f_u = 0 \tag{A14}$$

$$\epsilon^1 \;:\; P_{0_x}(g_u + u_1 f_{uu}) + P_{1_x} f_u = 0 \tag{A15}$$

$$\epsilon^2 \;:\; P_{0_x}(u_1 g_{uu} + \tfrac{1}{2}u_1^2 f_{uuu} + u_2 f_{uu}) + P_{1_x}(g_u + u_1 f_{uu}) + P_{2_x} f_u = 0 \tag{A16}$$

Now by using Eqs. (5), (A5–A12), and (A14–A16), the following equation determines the R_j:

$$R_j = -\sum_{n=0}^{j-1} P_{n_x}(g_{j-n-1} + f_{j-n}), \qquad j = 1,2,\ldots \tag{A17}$$

R_1 can now be found as

$$R_1 = -P_{0_x}(g_0 + f_1) = -P_{0_x}(g + u_1 f_u) \tag{A18}$$

where by using Eq. (A14), Eq. (A18) becomes

$$R_1 = -P_{0_x} g \tag{A19}$$

Similarly, R_2 is

$$R_2 = -P_{0_x}(g_1 + f_2) - P_{1_x}(g_0 + f_1) \tag{A20}$$

where by using Eqs. (A5), (A6), (A9), (A10), (A14), and (A15), Eq. (A20) becomes

$$R_2 = \frac{u_1^2}{2} P_{0_x} f_{uu} - P_{1_x} g \tag{A21}$$

Finally, R_3 is

$$R_3 = -P_{0_x}(g_2 + f_3) - P_{1_x}(g_1 + f_2) - P_{2_x}(g_0 + f_1) \tag{A22}$$

where by again using Eqs. (A5–A7), (A9–A11), and (A14–A16), Eq. (A22) becomes

$$R_3 = P_{0_x}\left(\frac{u_1^3}{3} f_{uuu} + u_1u_2 f_{uu} - \frac{u_1^2}{2} g_{uu}\right) + P_{1_x}\left(\frac{u_1^2}{2} f_{uu}\right) - P_{2_x} g$$
$$= P_{0_x}\left(\frac{u_1^3}{12} f_{uuu} + \frac{u_1u_2}{2} f_{uu}\right) - P_{1_x}\left(\frac{u_1}{2} g_u\right) - P_{2_x}\left(g + \frac{u_1}{2} f_u\right) \tag{A23}$$

Note that the preceding equations are for the general case where the perturbation dynamics, $g(y,u)$ are a function of the control. For the problem in this paper, the perturbation dynamics, $g(y)$, are only functions of the states. This leads to great simplification of the preceding equations. Specifically, the partials of g with respect to the control are equal to zero. This yields the following trivial form for Eqs. (A2) and (A4):

$$g(y,u^{\text{opt}}) = g(x) = g_0 \tag{A24}$$

As a result, Eqs. (A10–A12) are identically zero and Eq. (A13) becomes

$$P_x[f_u] = \left(\sum_{j=0}^{\infty} P_{j_x}\epsilon^j\right)\left(\sum_{j=0}^{\infty} P_{j_u}\epsilon^j\right) = 0 \tag{A25}$$

and Eqs. (A14–A16) become

$$\epsilon^0 \;:\; P_{0_x} f_u = 0 \tag{A26}$$

$$\epsilon^1 \;:\; P_{0_x} u_1 f_{uu} + P_{1_x} f_u = 0 \tag{A27}$$

$$\epsilon^2 \;:\; P_{0_x}(\tfrac{1}{2}u_1^2 f_{uuu} + u_2 f_{uu}) + P_{1_x} u_1 f_{uu} + P_{2_x} f_u = 0 \tag{A28}$$

Also, while Eqs. (A19) and (A21) remain unchanged, Eq. (A23) becomes

$$R_3 = P_{0_x}\left[\frac{u_1^3}{3} f_{uuu} + u_1u_2 f_{uu}\right] + P_{1_x}\left[\frac{u_1^2}{2} f_{uu}\right] - P_{2_x} g$$
$$= P_{0_x}\left[\frac{u_1^3}{12} f_{uuu} + \frac{u_1u_2}{2} f_{uu}\right] - P_{2_x}\left[g + \frac{u_1^2}{2} f_u\right] \tag{A29}$$

Appendix B: Solution of the Zeroth-Order Problem

As stated in the body of this paper and in Appendix A, the presence of the control in the perturbation dynamics complicates the optimal control expansion. However, the control can be removed from the perturbation dynamics by introducing a new independent variable z. If we let z be the new independent variable and defined such that the following relationship holds,

$$\frac{dz}{d\tau} = \frac{\beta}{wV}$$

then the dynamical equations become

$$\frac{dw}{dz} = -\gamma \tag{B1}$$

$$\frac{d\nu}{dz} = \frac{-(1 + \sigma^2 + \delta^2)}{E^*} - \epsilon\left(\frac{r_s(2e^{-\nu} - 1)\gamma}{wr}\right) \tag{B2}$$

$$\frac{d\gamma}{dz} = \delta + \epsilon\left(\frac{r_s(1 - e^{-\nu})}{wr}\right) \tag{B3}$$

$$\frac{d\psi}{dz} = \sigma + \epsilon\left(\frac{r_s \cos\psi \tan\phi}{wr}\right) \tag{B4}$$

where neither the new independent variable z or the controls σ or δ are explicitly present in the perturbation dynamics. This allows the use of the simpler optimal control expansion developed in Ref. 6. It also simplifies the zeroth-order control problem.

The Hamiltonian for this new problem, where $\epsilon = 0$, is

$$H = P_\psi\sigma - P_w\gamma - P_\nu\frac{(1 + \sigma^2 + \delta^2)}{E^*} + P_\gamma\delta \tag{B5}$$

where P_ψ, P_w, P_ν, and P_γ are the Lagrange multipliers.[7] For notational simplicity, the subscript $(\cdot)_0$, denoting the zeroth-order case, is dropped in this section. The augmented end-point function depends upon whether the final flight-path angle γ_f is fixed or free. However, first we will investigate the differential forms of P_ψ, P_w, P_ν, and P_γ given by the Hamiltonian:

$$\frac{dP_\psi}{dz} = -H_\psi = 0 \Rightarrow P_\psi = \text{const} \tag{B6}$$

$$\frac{dP_w}{dz} = -H_w = 0 \Rightarrow P_w = \text{const} \tag{B7}$$

$$\frac{dP_\nu}{dz} = -H_\nu = 0 \Rightarrow P_\nu = \text{const} \tag{B8}$$

$$\frac{dP_\gamma}{dz} = -H_\gamma = P_w \Rightarrow P_\gamma = P_w z + \text{const} \tag{B9}$$

Here, we will only look at the case for γ_f-fixed corresponding to the end-point function

$$G = -\nu_f + \nu_\psi(\psi_f - \psi_{f_s}) + \nu_w(w_f - w_{f_s}) + \nu_\gamma(\gamma_f - \gamma_{f_s}) \tag{B10}$$

where $(\cdot)_f$ indicates the actual final value of $(\cdot)$, $(\cdot)_{f_s}$ indicates the specified terminal value of $(\cdot)$, and ν_ψ, ν_w, and ν_γ are Lagrange multipliers. Now the natural boundary conditions are given by

$$H_f = -G_{z_f} = 0 \tag{B11}$$

$$P_{\psi_f} = G_{\psi_f} = \nu_\psi \tag{B12}$$

$$P_{w_f} = G_{w_f} = \nu_w \tag{B13}$$

$$P_{\nu_f} = G_{\nu_f} = -1 \tag{B14}$$

$$P_{\gamma_f} = G_{\gamma_f} = \nu_\gamma \tag{B15}$$

These equations along with Eqs. (B6–B9) imply that

$$P_\psi = \nu_\psi \tag{B16}$$

$$P_w = \nu_w \tag{B17}$$

$$P_\nu = -1 \tag{B18}$$

$$P_\gamma = P_w z + C \tag{B19}$$

where ν_ψ, ν_w, and $C = \nu_\gamma - P_w z_f$ are constants.

The optimal control relationships for this problem can now be determined using the optimality conditions:

$$H_\delta = P_\gamma - \frac{2\delta P_\nu}{E^*} = 0 \tag{B20}$$

$$H_\sigma = P_\psi - \frac{2\sigma P_\nu}{E^*} = 0 \tag{B21}$$

Equations (B20) and (B21) now yield the following optimal lift and side force control relationships:

$$\delta = \frac{E^* P_\gamma}{2P_\nu} \tag{B22}$$

$$\sigma = \frac{E^* P_\psi}{2P_\nu} \tag{B23}$$

Note that in this formulation, Eqs. (B1–B4), the controls do not appear in the perturbation dynamics. This implies that the optimal control relationships, Eqs. (B22) and (B23), are the same for the zeroth-order problem and for the unreduced problem which includes the perturbation dynamics. The difference between the two lies in the form and values of the Lagrange multipliers P_γ, P_ν, and P_ψ. For the zeroth-order control problem, Eqs. (B16–B19) imply that Eqs. (B22) and (B23) can be written as

$$\delta = \frac{-E^* P_\gamma}{2} \tag{B24}$$

$$\sigma = \frac{-E^* P_\psi}{2} = \text{const} \tag{B25}$$

Now since σ is a constant for the zeroth-order optimal control problem, Eq. (B4) can be solved to obtain

$$\psi = \sigma z + \text{const} \tag{B26}$$

Since the only constraint put on z is that it satisfies Eq. (B4), an additional constraint must be placed on z in order to solve Eq. (B26). Therefore, we assume that $z_i = 0$, resulting in the following equation:

$$\psi = \sigma z + \psi_i \quad \text{or} \quad z = \frac{\Delta\psi}{\sigma} \tag{B27}$$

where $\Delta\psi = \psi - \psi_i$. Also, since σ is a constant and ψ can be shown to be monotonic, then z is also monotonic along the zeroth-order optimal trajectory.

Now note from Eq. (B5) that H is not explicitly a function of z. This implies that H is a constant along the zeroth-order optimal trajectory and Eq. (B11) implies that $H = 0$, which yields

$$P_\psi\sigma - P_w\gamma + \frac{(1 + \sigma^2 + \delta^2)}{E^*} + P_\gamma\delta = 0 \tag{B28}$$

By using Eqs. (B24) and (B25), Eq. (B28) can be further reduced to

$$\sigma^2 = 1 - E^* P_w\gamma - \frac{E^{*2} P_\gamma^2}{4} \tag{B29}$$

Now Eqs. (B1–B4) can be solved for the state relationships in terms of z using Eqs. (B19), (B24), (B25), and (B29). This results in the following equations:

$$\psi = \sigma z + \psi_i \tag{B30}$$

$$\gamma = \frac{-E^* P_w}{4} z^2 - \frac{E^* C}{2} z + \gamma_i \tag{B31}$$

$$w = \frac{E^* P_w}{12} z^3 + \frac{E^* C}{4} z^2 - \gamma_i z + w_i \tag{B32}$$

$$v = -P_w(w - w_i) - \frac{2z}{E^*} + v_i \tag{B33}$$

where

$$P_w = \frac{-24}{E^* z_f^2}\left(\frac{\Delta w}{z_f} + \frac{(\gamma_i + \gamma_f)}{2}\right) \tag{B34}$$

$$C = -\left(\frac{2\Delta\gamma}{E^* z_f} + \frac{P_w z_f}{2}\right) \tag{B35}$$

with boundary conditions

$$\psi_i,\ \gamma_i,\ w_i, \text{ and } v_i \equiv \text{given}, \qquad z_i = 0 \tag{B36}$$

$$\psi_f,\ \gamma_f, \text{ and } w_f \equiv \text{given} \tag{B37}$$

and $\Delta w = w_f - w_i$, $\Delta\psi = \psi_f - \psi_i$, and $\Delta\gamma = \gamma_f - \gamma_i$.

Now Eqs. (B19), (B30), (B34), and (B35) can be substituted into Eq. (B29) along with some algebraic manipulation to yield the following fourth-order polynomial in σ:

$$A_4\sigma^4 + A_3\sigma^3 + A_2\sigma^2 - A_0 = 0 \tag{B38}$$

where

$$A_4 = 36(\Delta w)^2 \tag{B39}$$

$$A_3 = 24(\gamma_f + \gamma_i)\Delta w \Delta\psi \tag{B40}$$

$$A_2 = [(\Delta\psi)^2 + (\Delta\gamma)^2 + 3(\gamma_f + \gamma_i)^2](\Delta\psi)^2 \tag{B41}$$

$$A_0 = (\Delta\psi)^4 \tag{B42}$$

From Eqs. (B38–B42), the solution for the zeroth-order optimal control σ for the γ_f-fixed case can be found. The same process can be used to solve the γ_f-free case, which also results in a fourth-order polynomial in σ.

The optimal lift control δ can also be determined by using Eqs. (B19) and (B24) to yield the relationship:

$$\delta = \frac{-E^*}{2}(P_w z + C) \tag{B43}$$

where P_w and C are given by Eqs. (B34) and (B35). The zeroth-order lift coefficient λ and bankangle μ are now given by

$$\lambda = C_L/C_L^* = \sqrt{(\sigma^2 + \delta^2)} \tag{B44}$$

$$\mu = \tan^{-1}\left(\frac{\sigma}{\delta}\right) \tag{B45}$$

Also the zeroth-order optimal cost can be written as

$$J = -v_f = P_w(\Delta w) + \frac{2z_f}{E^*} - v_i \tag{B46}$$

The apparent benefit of creating this new variable z is to simplify the equations and remove the control from the perturbation dynamics. However, an even more important benefit comes from the fact that by making this transformation the "exact" control relationships, Eqs. (B22) and (B23), are of a simple form that happens to be of the same form as the zeroth-order control relationships. All of these factors make it easier to find higher-order correction terms.

Appendix C: First-Order Correction Terms

The first-order correction term is the coefficient multiplying the small parameter ϵ in the optimal control expansion. This term along with the zeroth-order optimal control solution previously derived yield a first-order corrected optimal control relationship. However, the form of the first-order correction term depends upon the form of the zeroth-order optimal control. Therefore, the method by which the zeroth-order optimal control solution is obtained is important. See Appendix B for the solution and discussion of the zeroth-order optimal control problem.

The optimal control relationships for this problem are given by

$$\delta = \frac{E^* P_\gamma}{2P_v} \tag{C1}$$

$$\sigma = \frac{E^* P_\psi}{2P_v} \tag{C2}$$

where P_γ, P_ψ, and P_v are Lagrange multipliers for the optimal control problem. The Lagrange multipliers can be expanded in the small parameter ϵ as follows:

$$P_\gamma = P_{0_\gamma} + P_{1_\gamma}\epsilon + P_{2_\gamma}\epsilon^2 + P_{3_\gamma}\epsilon^3 + \ldots \tag{C3}$$

$$P_\psi = P_{0_\psi} + P_{1_\psi}\epsilon + P_{2_\psi}\epsilon^2 + P_{3_\psi}\epsilon^3 + \ldots \tag{C4}$$

$$P_v = P_{0_v} + P_{1_v}\epsilon + P_{2_v}\epsilon^2 + P_{3_v}\epsilon^3 + \ldots \tag{C5}$$

Similarly, the controls σ and δ can also be expanded

$$\sigma = \sigma_0 + \sigma_1\epsilon + \sigma_2\epsilon^2 + \sigma_3\epsilon^3 + \ldots \tag{C6}$$

$$\delta = \delta_0 + \delta_1\epsilon + \delta_2\epsilon^2 + \delta_3\epsilon^3 + \ldots \tag{C7}$$

Substituting the previous expansions into the control relationships of Eq. (17) and equating terms of equal powers of ϵ yields the following feedback control expansion terms:

$$\sigma_0 = \frac{E^* P_{0_\psi}}{2P_{0_v}} \tag{C8}$$

$$\sigma_1 = \sigma_0\left(\frac{P_{1_\psi}}{P_{0_\psi}} - \frac{P_{1_v}}{P_{0_v}}\right) \tag{C9}$$

$$\sigma_2 = \sigma_0\left(\frac{P_{2_\psi}}{P_{0_\psi}} - \frac{P_{2_v}}{P_{0_v}}\right) - \sigma_1\left(\frac{P_{1_v}}{P_{0_v}}\right) \tag{C10}$$

$$\vdots$$

$$\delta_0 = \frac{E^* P_{0_\gamma}}{2P_{0_v}} \tag{C11}$$

$$\delta_1 = \delta_0\left(\frac{P_{1_\gamma}}{P_{0_\gamma}} - \frac{P_{1_v}}{P_{0_v}}\right) \tag{C12}$$

$$\delta_2 = \delta_0\left(\frac{P_{2_\gamma}}{P_{0_\gamma}} - \frac{P_{2_v}}{P_{0_v}}\right) - \delta_1\left(\frac{P_{1_v}}{P_{0_v}}\right) \tag{C13}$$

$$\vdots$$

which are of the same form as in Ref. 6.

P_{0_γ}, P_{0_ψ}, P_{0_ν}, and P_{0_w} are given by the solution of the zeroth-order optimal control problem (see Appendix B) to be

$$P_{0_\gamma} = P_{0_w} z + C \tag{C14}$$

$$P_{0_\psi} = \frac{-2\sigma_0}{E^*} = \text{const} \tag{C15}$$

$$P_{0_\nu} = -1 \tag{C16}$$

$$P_{0_w} = \frac{-24}{E^* z_f^2}\left(\frac{\Delta w}{z_f} + \frac{(\gamma_i + \gamma_f)}{2}\right) \tag{C17}$$

where

$$C = \frac{2}{E^* z_f^2}\left(\frac{6\Delta w}{\Delta \psi} + 3\,(\gamma_i + \gamma_f) - \Delta\gamma\right) \tag{C18}$$

As given in Eq. (7) and in Ref. 6, the general equation for the expansion terms of the Lagrange multipliers is given as

$$P_{j_x} = \frac{\partial P_j}{\partial x} = -\int_t^{t_f} \frac{\partial R_j}{\partial x}\, d\tau - R_j|_{t_f} \frac{\partial t_f}{\partial x} \tag{C19}$$

where j refers to the jth-order term, x refers to a current (or initial) state, the integration is along the zeroth-order solution, and R_j is given in Appendix A by

$$R_j = -\sum_{n=0}^{j-1} P_{n_x}(g_{j-n-1} + f_{j-n}), \qquad j = 1,2,\ldots \tag{C20}$$

where f and g are the primary and perturbation dynamics, respectively.

Therefore, for the first-order case Eq. (A19),

$$R_1 = -P_{0_x}\, g(y) \tag{C21}$$

or

$$R_1 = -P_{0_\nu}\left(\frac{r_0(1-2e^{-\nu})\gamma}{wr}\right) - P_{0_\gamma}\frac{r_0(1-e^{-\nu})}{wr} \tag{C22}$$

Now, substituting Eqs. (C14) and (C16) into (C22) yields

$$R_1 = \frac{r_0}{wr}\,[\gamma(1-2e^{-\nu}) - (P_{0_w} z + C)(1-e^{-\nu})] \tag{C23}$$

This gives the following equation:

$$P_1 = -\int_0^{z_f} \frac{r_0}{w' r'}\,[\gamma'(1-2e^{-\nu'}) - (P_{0_w} z' + C)(1-e^{-\nu'})]\, dz' \tag{C24}$$

where $(\cdot)'$ indicates integral variables computed along the zeroth-order solution.

Now that we have formulated the integral equation for the first-order term in the cost expansion, all that is left is the nontrivial task of taking the appropriate partial derivatives needed in the following Lagrange multiplier integral equation:

$$P_{1_\psi} = \frac{\partial P_1}{\partial \psi_i} = -\int_0^{z_f} \frac{\partial R_1}{\partial \psi_i}\, dz' - R_1|_{z_f}\frac{\partial z_f}{\partial \psi_i} \tag{C25}$$

$$P_{1_\gamma} = \frac{\partial P_1}{\partial \gamma_i} = -\int_0^{z_f} \frac{\partial R_1}{\partial \gamma_i}\, dz' - R_1|_{z_f}\frac{\partial z_f}{\partial \gamma_i} \tag{C26}$$

$$P_{1_\nu} = \frac{\partial P_1}{\partial \nu_i} = -\int_0^{z_f} \frac{\partial R_1}{\partial \nu_i}\, dz' \tag{C27}$$

where $\partial z_f/\partial \nu_i = 0$ since σ_0 is not a function of ν_i, as shown in Eqs. (B38–B42). The partials of R_1 are

$$\frac{\partial R_1}{\partial \psi_i} = \frac{r_0}{wr}\left[(1-2e^{-\nu})\frac{\partial\gamma}{\partial\psi_i} + \frac{2(1-e^{-\nu})}{E^*}\frac{\partial\delta_0}{\partial\psi_i} + 2\left(\gamma + \frac{\delta_0}{E^*}\right)e^{-\nu}\frac{\partial\nu}{\partial\psi_i}\right] + \frac{R_1}{w}\left(\frac{\beta}{r} - 1\right)\frac{\partial w}{\partial \psi_i} \tag{C28}$$

$$\frac{\partial R_1}{\partial \gamma_i} = \frac{r_0}{wr}\left[(1-2e^{-\nu})\frac{\partial\gamma}{\partial\gamma_i} + \frac{2(1-e^{-\nu})}{E^*}\frac{\partial\delta_0}{\partial\gamma_i} + 2\left(\gamma + \frac{\delta_0}{E^*}\right)e^{-\nu}\frac{\partial\nu}{\partial\gamma_i}\right] + \frac{R_1}{w}\left(\frac{\beta}{r} - 1\right)\frac{\partial w}{\partial \gamma_i} \tag{C29}$$

$$\frac{\partial R_1}{\partial \nu_i} = \frac{2r_0}{wr}\left(\gamma + \frac{\delta_0}{E^*}\right)e^{-\nu} \tag{C30}$$

Note, the preceding partials are dependent upon the following partials:

$$\frac{\partial \gamma}{\partial \psi_i} = \frac{-E^* z}{2}\left(\frac{z}{2}\frac{\partial P_{0_w}}{\partial \psi_i} + \frac{\partial C}{\partial \psi_i}\right) \tag{C31}$$

$$\frac{\partial \gamma}{\partial \gamma_i} = 1 - \frac{-E^* z}{2}\left(\frac{z}{2}\frac{\partial P_{0_w}}{\partial \gamma_i} + \frac{\partial C}{\partial \gamma_i}\right) \tag{C32}$$

$$\frac{\partial \delta_0}{\partial \psi_i} = \frac{-E^*}{2}\left(z\frac{\partial P_{0_w}}{\partial \psi_i} + \frac{\partial C}{\partial \psi_i}\right) \tag{C33}$$

$$\frac{\partial \delta_0}{\partial \gamma_i} = \frac{-E^*}{2}\left(z\frac{\partial P_{0_w}}{\partial \gamma_i} + \frac{\partial C}{\partial \gamma_i}\right) \tag{C34}$$

$$\frac{\partial \nu}{\partial \psi_i} = -P_{0_w}\frac{\partial w}{\partial \psi_i} - (w - w_i)\frac{\partial P_{0_w}}{\partial \psi_i} \tag{C35}$$

$$\frac{\partial \nu}{\partial \gamma_i} = -P_{0_w}\frac{\partial w}{\partial \gamma_i} - (w - w_i)\frac{\partial P_{0_w}}{\partial \gamma_i} \tag{C36}$$

$$\frac{\partial w}{\partial \psi_i} = \frac{E^* z^2}{4}\left(\frac{z}{3}\frac{\partial P_{0_w}}{\partial \psi_i} + \frac{\partial C}{\partial \psi_i}\right) \tag{C37}$$

$$\frac{\partial w}{\partial \gamma_i} = \frac{E^* z^2}{4}\left(\frac{z}{3}\frac{\partial P_{0_w}}{\partial \gamma_i} + \frac{\partial C}{\partial \gamma_i}\right) - z \tag{C38}$$

$$\frac{\partial z_f}{\partial \psi_i} = \frac{-1}{\sigma_0}\left(1 + z_f\frac{\partial \sigma_0}{\partial \psi_i}\right) \tag{C39}$$

$$\frac{\partial z_f}{\partial \gamma_i} = \frac{-z_f}{\sigma_0}\frac{\partial \sigma_0}{\partial \gamma_i} \tag{C40}$$

where Eq. (B27) is used to determine Eq. (C39) and (C40). Again, the previous equations are dependent upon the following additional partials:

$$\frac{\partial P_{0_w}}{\partial \psi_i} = \frac{12}{E^* z_f^3}\left(\frac{6\Delta w}{z_f} + 2(\gamma_i + \gamma_f)\right)\frac{\partial z_f}{\partial \psi_i} \tag{C41}$$

$$\frac{\partial P_{0_w}}{\partial \gamma_i} = \frac{12}{E^* z_f^3}\left(\frac{6\Delta w}{z_f} + 2(\gamma_i + \gamma_f)\right)\frac{\partial z_f}{\partial \gamma_i} - \frac{12}{E^* z_f^2} \tag{C42}$$

$$\frac{\partial C}{\partial \psi_i} = \left(\frac{2\Delta\gamma}{E^* z_f^2} - \frac{P_w}{2}\right)\frac{\partial z_f}{\partial \psi_i} - \frac{z_f}{2}\frac{\partial P_{0_w}}{\partial \psi_i} \tag{C43}$$

$$\frac{\partial C}{\partial \gamma_i} = \left(\frac{2\Delta\gamma}{E^* z_f^2} - \frac{P_w}{2}\right)\frac{\partial z_f}{\partial \gamma_i} - \frac{z_f}{2}\frac{\partial P_{0_w}}{\partial \gamma_i} + \frac{2}{E^* z_f} \tag{C44}$$

$$\frac{\partial \sigma_0}{\partial \psi} = \frac{\left[24(\gamma_f+\gamma_i)\Delta w \sigma_0^3 + 2\left(\Delta\psi^3 + \frac{A_2}{\Delta\psi}\right)\sigma_0^2 - 4(\Delta\psi)^3\right]}{\sigma_0(4A_4\sigma_0^2 + 3A_3\sigma_0 + 2A_2)} \tag{C45}$$

$$\frac{\partial \sigma_0}{\partial \gamma} = \frac{-[24\Delta w \Delta\psi \sigma_0 + (4\gamma_f + 8\gamma_i)(\Delta\psi)^2]\sigma_0}{(4A_4\sigma_0^2 + 3A_3\sigma_0 + 2A_2)} \tag{C46}$$

where the coefficients, A_4, A_3, A_2, and A_0, of the fourth-order polynomial solution to the zeroth-order control problem, Eq. (B38), are derived in Appendix B and given in Eqs. (B39–B42) and in the main body of the paper.

The last set of partials are only functions of known constants or state variables. As a result, these and the preceding partials can all be calculated in the numerical solution of the Lagrange multipliers given by the integrals of Eqs. (C25–C27). These can in turn be used in Eqs. (C9) and (C12) to find the first-order correction coefficients in Eqs. (C6) and (C7). Thus, we have a first-order corrected optimal control solution.

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